

A CONVEX STOCHASTIC OPTIMIZATION PROBLEM ARISING FROM PORTFOLIO SELECTION

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A continuous-time financial portfolio selection model with expected utility maximization typically boils down to solving a (static) convex stochastic optimization problem in terms of the terminal wealth, with a budget constraint. In literature the latter is solved by assuming *a priori* that the problem is well-posed (i.e., the supremum value is finite) and a Lagrange multiplier exists (and as a consequence the optimal solution is attainable). In this paper it is first shown, via various counter-examples, neither of these two assumptions needs to hold, and an optimal solution does not necessarily exist. These anomalies in turn have important interpretations in and impacts on the portfolio selection modeling and solutions. Relations among the non-existence of the Lagrange multiplier, the ill-posedness of the problem, and the non-attainability of an optimal solution are then investigated. Finally, explicit and easily verifiable conditions are derived which lead to finding the unique optimal solution.

KEY WORDS: portfolio selection, convex stochastic optimization, Lagrange multiplier, well-posedness, attainability

1 INTRODUCTION

Given a probability space (Ω, \mathcal{F}, P) , consider the following constrained stochastic optimization problem

$$(1.1) \quad \begin{aligned} &\text{Maximize} && Eu(X) \\ &\text{subject to} && E[X\xi] = a, \quad X \geq 0 \text{ is a random variable,} \end{aligned}$$

where $a > 0$ is a parameter, $\xi > 0$ a given scalar-valued random variable, $u(\cdot): \mathbb{R}^+ \mapsto \mathbb{R}^+$ a twice differentiable, strictly increasing, strictly concave function with $u(0) = 0, u'(0+) = +\infty, u'(+\infty) = 0$. Define $V(a) = \sup_{E[X\xi]=a, X \geq 0 \text{ is a r.v.}} Eu(X)$.

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It is well known that many continuous-time financial portfolio selection problems with expected utility maximization boil down to solving problem (1.1). In the context of a portfolio model, $u(\cdot)$ is the utility function (all the assumed properties on $u(\cdot)$ have economic interpretations), ξ is the so-called pricing kernel or state price density, a is the initial wealth (hence the first constraint is the budget constraint), and X is the terminal wealth to be determined. Once an optimal X^* to (1.1) is found, the portfolio replicating X^* is the optimal portfolio for the original dynamic portfolio choice problem, if the market is complete. For details see, e.g., Cvitanic and Karatzas (1992), Karatzas (1997), Karatzas and Shreve (1998), Korn (1997).

In literature (1.1) is usually solved by the Lagrange method, which is summarized in the following theorem.

THEOREM 1.1 *If (1.1) admits an optimal solution X^* whose objective value is finite, then there exists $\lambda > 0$ such that $X^* = (u')^{-1}(\lambda\xi)$. Conversely, if $E[(u')^{-1}(\lambda\xi)\xi] = a < +\infty$ and $E[u((u')^{-1}(\lambda\xi))] < +\infty$, then $X^* = (u')^{-1}(\lambda\xi)$ is optimal for (1.1) with parameter a .*

This theorem provides an efficient scheme to find the optimal solution for Problem (1.1): For any $a > 0$, solve the Lagrange equation $E[(u')^{-1}(\lambda\xi)\xi] = a$ – if one could – to determine a Lagrange multiplier λ , and then $X^* = (u')^{-1}(\lambda\xi)$ is the optimal (automatically unique as the utility function is strictly concave) solution for (1.1), if $Eu(X^*)$ is finite.

However, there are many issues about Problem (1.1) that are left untouched by the preceding theorem/scheme. To elaborate, in general there are the following progressive issues related to an optimization problem such as (1.1):

- *Feasibility*: whether there is at least one solution satisfying all the constraints involved. For (1.1), since $X = a/\xi$ is a feasible solution, the feasibility is not an issue.¹
- *Well-posedness*: whether the supremum value of the problem with a non-empty feasible set is finite (in which case the problem is called well-posed) or $+\infty$ (ill-posed). An ill-posed problem is a mis-formulated one: the trade-off is not set right so one could always push the objective value to be arbitrarily high.²
- *Attainability*: whether a well-posed problem admits an optimal solution. It may or may not.
- *Uniqueness*: whether an attainable problem has a unique optimal solution. It is not an issue for (1.1), since uniqueness holds automatically due to the strict concavity of the utility function.

Clearly, Theorem 1.1 covers only the case when the problem is well-posed and the attainability holds, by assuming *a priori* that a Lagrange multiplier exists (indeed, in the context of portfolio

¹Feasibility could be by itself an interesting problem if more complex constraints are involved. See Section 3 of Bielecki et. al (2005) for an example.

²Again, well-posedness is an important, sometimes *very* difficult, problem in its own right; see Jin and Zhou (2006) for a behavioral portfolio selection model where the well-posedness becomes an eminent issue. Also see Korn and Kraft (2004) for more ill-posed examples.

selection the existing work always *assumes* that the Lagrange multiplier exists; see Theorem 2.2.2 in page 7 of Karatzas (1997) and page 65 of Korn (1997)³. Moreover, in Theorem 2.2.2 in page 7 of Karatzas (1997) and Assumption 6.2 in page 773 of Cvitanic and Karatzas (1992), it is assumed up front that the underlying problem is well-posed.⁴ In this paper we will first show, through various counter-examples, that none of the aforementioned assumptions that have all along been taken for granted needs to hold true. Then, we will address the following questions: When does the Lagrange multiplier exist? What if it does not? What does it have to do with the well-posedness and attainability? What are the conditions ensuring the existence of a unique optimal solution for (1.1) for a given $a > 0$ or for any $a > 0$?

The aim of this paper is to give a thorough treatment of (1.1), including answers to the above questions. In particular, Section 2 reveals the possibility of non-existence of the Lagrange multiplier. Section 3 studies the implications of the non-existence of the Lagrange multiplier, and Section 4 shows the possibility of ill-posedness even with the existence of the Lagrange multiplier. Finally, Section 5 presents easily verifiable conditions for uniquely solving (1.1).

2 NON-EXISTENCE OF LAGRANGE MULTIPLIER

It is possible that the Lagrange multiplier simply does not exist, which will be demonstrated in this section via several examples.

First off, define

$$(2.1) \quad f(\lambda) = E[(u')^{-1}(\lambda\xi)\xi], \quad \lambda > 0.$$

Then $f(\cdot)$ is non-increasing (notice that $f(\cdot)$ may take value $+\infty$). The following lemma is evident given the monotonicity of $(u')^{-1}(\cdot)$ and the monotone convergence theorem.

LEMMA 2.1 *If $f(\lambda_0) < +\infty$ for some $\lambda_0 > 0$, then $f(\cdot)$ is continuous on $(\lambda_0, +\infty)$ and right continuous at λ_0 , with $f(+\infty) = 0$.*

It follows from Lemma 2.1 that if $f(\lambda_0) < +\infty$ for some $\lambda_0 > 0$, then the Lagrange multiplier exists for any $0 < a \leq a_0 := E[(u')^{-1}(\lambda_0\xi)\xi]$. In particular, if

$$(2.2) \quad f(\lambda) < +\infty \quad \forall \lambda > 0,$$

then the Lagrange multiplier exists for any $a > 0$. This is why in existing literature (2.2) is usually assumed up front (see, e.g., Karatzas (1997), p. 37, (2.2.11) and Korn (1997), p. 65, (24)). Now, we are to show that this assumption may not hold even for simple cases.

³In these references it is assumed that $f(\lambda) = E[(u')^{-1}(\lambda\xi)\xi] < +\infty$ for any $\lambda > 0$, which is equivalent to the existence of the Lagrange multiplier for any $a > 0$; see Section 2 for details.

⁴Some of the references cited here deal with models with consumptions; yet the essence of the Lagrange method remains the same.

EXAMPLE 2.1 Take $u(x) = \sqrt{x}$, $x \geq 0$, $P(\xi \leq t) = 1 - e^{-t}$, $t \geq 0$. In this example, $u'(x) = \frac{1}{2\sqrt{x}}$, $(u')^{-1}(y) = (2y)^{-2}$, and $f(\lambda) = E[(u')^{-1}(\lambda\xi)\xi] = \frac{1}{4\lambda^2}E\xi^{-1} = +\infty$ for any $\lambda > 0$. Therefore $E[(u')^{-1}(\lambda\xi)\xi] = a$ admits no solution for any $a > 0$.

In the above example the Lagrange multiplier does not exist for *any* $a > 0$. In the following examples, Lagrange multipliers exist for *some* $a > 0$, and do not for *other* $a > 0$.

EXAMPLE 2.2 Define $p(x) = \frac{e^x - 1 - x - x^2/2! - x^3/3!}{x^2} = \sum_{n=2}^{+\infty} \frac{x^n}{(n+2)!}$, $g(x) = p(\frac{1}{x})$, $h(x) = g^{-1}(x)$, $x > 0$. Take

$$u(x) = \begin{cases} xh(x) + \int_0^{1/h(x)} \frac{p(y)}{y^2} dy, & x > 0, \\ 0, & x = 0, \end{cases}$$

and $P(\xi \geq t) = 1 - e^{-1/t}$, $t > 0$; or $1/\xi$ follows the exponential distribution with parameter 1.

In this example, $p(\cdot)$ is strictly increasing with $p(0+) = 0$, $p(+\infty) = +\infty$; hence $g(\cdot)$ is strictly decreasing with $g(0+) = +\infty$, $g(+\infty) = 0$, and $h(\cdot)$ is well-defined and strictly decreasing with $h(0+) = +\infty$, $h(+\infty) = 0$. All these functions are smooth.

For the utility function $u(\cdot)$, notice that $\int_0^x \frac{p(y)}{y^2} dy = \int_0^x \sum_{n=0}^{\infty} \frac{y^n}{(n+4)!} dy = \sum_{n=0}^{+\infty} \frac{x^{n+1}}{(n+4)!(n+1)}$ is well-defined for any $x > 0$, and

$$\lim_{x \rightarrow 0+} xh(x) = \lim_{y \rightarrow +\infty} g(y)y = \lim_{y \rightarrow +\infty} p(\frac{1}{y})y = 0,$$

which means that $u(\cdot)$ is right-continuous at 0. Furthermore, for any $x > 0$

$$\begin{aligned} u'(x) &= h(x) + xh'(x) - \frac{p(1/h(x))}{1/h(x)^2} \frac{h'(x)}{h(x)^2} \\ &= h(x) + xh'(x) - p(1/h(x))h'(x) \\ &= h(x) + xh'(x) - g(h(x))h'(x) \\ &= h(x). \end{aligned}$$

Therefore $u(\cdot)$ is concave and $u'(0+) = h(0+) = +\infty$, $u'(+\infty) = h(+\infty) = 0$. Moreover, $u'(x) = h(x)$, and $(u')^{-1}(y) = g(y) = \sum_{n=2}^{+\infty} \frac{1}{(n+2)!y^n}$. On the other hand, from the distribution of ξ it follows easily that $E\xi^{-n} = n!$ for any $n \in \mathbb{N}$.

Now let us calculate $f(\lambda) = E[(u')^{-1}(\lambda\xi)\xi]$ for any $\lambda > 0$:

$$\begin{aligned} f(\lambda) &= E[g(\lambda\xi)\xi] \\ &= E\left[\sum_{n=2}^{+\infty} \frac{1}{(n+2)! \lambda^n} \xi^{-(n-1)}\right] \\ &= \sum_{n=2}^{+\infty} \frac{(n-1)!}{(n+2)! \lambda^n} \\ &= \sum_{n=2}^{+\infty} \frac{1}{(n+2)(n+1)n} \left(\frac{1}{\lambda}\right)^n. \end{aligned}$$

By the convergence of series, we know that $f(\lambda) < +\infty$ if and only if $\lambda \geq 1$.

Define $a_1 = f(1) = E[(u')^{-1}(\xi)\xi] = \sum_{n=2}^{+\infty} \frac{1}{(n+2)(n+1)n} = \frac{1}{12}$. Then for any $0 < a \leq a_1$, we can find a Lagrange multiplier $\lambda \geq 1$ such that $E[(u')^{-1}(\lambda\xi)\xi] = a$. On the other hand, the Lagrange multiplier is non-existent when $a > a_1$.

In the preceding examples ξ is related to the exponential distribution, whereas in applying to portfolio selection ξ is typically lognormal. The next example shows such a case.

EXAMPLE 2.3 Take a positive random variable ξ satisfying $0 < E[\xi^{-(n-1)}] < +\infty \forall n \geq 1$ and $\lim_{n \rightarrow +\infty} \frac{E[\xi^{-(n-1)}]}{E[\xi^{-n}]} = 0$ (e.g., when ξ is lognormal). Define $a_n = \frac{1}{n^2 E[\xi^{-(n-1)}]}$, $n \geq 2$, and $p(x) = \sum_{n=2}^{+\infty} a_n x^n$, $g(x) = p(\frac{1}{x})$, $h(x) = g^{-1}(x)$, $x > 0$. Take

$$u(x) = \begin{cases} xh(x) + \int_0^{1/h(x)} \frac{p(y)}{y^2} dy, & x > 0, \\ 0, & x = 0. \end{cases}$$

Exactly the same analysis as in Example 2.2 yields that $u(\cdot)$ is a utility function satisfying all the required conditions, with $u'(x) = h(x)$ and $(u')^{-1}(x) = g(x) = \sum_{n=2}^{+\infty} a_n x^{-n}$.

Now, for any $\lambda > 0$,

$$f(\lambda) = E[g(\lambda\xi)\xi] = E\left[\sum_{n=2}^{+\infty} a_n \lambda^{-n} \xi^{-(n-1)}\right] = \sum_{n=2}^{+\infty} \frac{1}{n^2 \lambda^n}.$$

Hence $f(\lambda) < +\infty$ if and only if $\lambda \geq 1$. As a result, the Lagrange multiplier exists if and only if $0 < a \leq a_1$, where $a_1 = f(1) = E[(u')^{-1}(\xi)\xi] = \sum_{n=2}^{+\infty} \frac{1}{n^2} = \frac{\pi^2-6}{6}$.

3 IMPLICATION OF NON-EXISTENCE OF LAGRANGE MULTIPLIER

So, if the Lagrange multiplier does not exist, what can we say about the underlying optimization problem (1.1)? Theorem 1.1 implies that the non-existence of the Lagrange multiplier is an indication of either the ill-posedness or the non-attainability of (1.1). In this section we elaborate on this.

THEOREM 3.1 *If $E[(u')^{-1}(\lambda\xi)\xi] = +\infty$ for any $\lambda > 0$, then $V(a) = +\infty$ for any $a > 0$.*

Proof: Fix $\lambda_0 > 0$ and $a > 0$. Since $E[(u')^{-1}(\lambda_0\xi)\xi] = +\infty$, one can find a set $A \in \mathcal{F}$ such that $E[(u')^{-1}(\lambda_0\xi)\xi \mathbf{1}_A] \in (a, +\infty)$. Define $h(\lambda) = E[(u')^{-1}(\lambda\xi)\xi \mathbf{1}_A]$, $\lambda \in [\lambda_0, +\infty)$. Then $h(\cdot)$ is non-increasing and continuous on $[\lambda_0, +\infty)$ with $h(+\infty) = 0$; hence there exists $\lambda_1 > \lambda_0$ such that $h(\lambda_1) = a$.

Denote $X_1 = (u')^{-1}(\lambda_1\xi)\mathbf{1}_A$, which is a feasible solution for Problem (1.1) with parameter a , and $V(a) \geq E[u(X_1)\mathbf{1}_A] \geq E[X_1 u'(X_1)\mathbf{1}_A] = E[(u')^{-1}(\lambda_1\xi)\lambda_1\xi \mathbf{1}_A] = \lambda_1 a > \lambda_0 a$. (Here we have used the fact that $u(x) \geq xu'(x) \forall x > 0$ owing to the concavity of $u(\cdot)$ and that $u(0) = 0$.) Since $\lambda_0 > 0$ is arbitrary, we arrive at $V(a) \geq \lim_{\lambda_0 \rightarrow +\infty} \lambda_0 a = +\infty$. \square

This theorem indicates that if the Lagrange multiplier does not exist for *all* $a > 0$, then (1.1) is ill-posed for *all* $a > 0$. Example 2.1 exemplifies such a case. Now, if the Lagrange multiplier does not exist for only *some* a (such as in Examples 2.2 and 2.3), is it still possible that (1.1) is well-posed for the same a ? To study this, we need the following lemma.

LEMMA 3.1 $V(a) < +\infty, \forall a > 0$ if and only if $\exists a > 0$ such that $V(a) < +\infty$.

Proof: It suffices to prove that if $V(a) < +\infty$ for some $a > 0$ then $V(b) < +\infty$ for any $b > 0$.

For $b \geq a$, we have

$$\begin{aligned} V(b) &= \sup_{E[X\xi]=b, X \geq 0} Eu(X) = \sup_{E[X\xi]=a, X \geq 0} Eu\left(\frac{b}{a}X\right) \\ &\leq \sup_{E[X\xi]=a, X \geq 0} \frac{b}{a}Eu(X) = \frac{b}{a}V(a) < +\infty, \end{aligned}$$

where the first inequality is due to the concavity of $u(\cdot)$ and $u(0) = 0$.

For any $0 < b < a$,

$$\begin{aligned} V(b) &= \sup_{E[X\xi]=b, X \geq 0} Eu(X) = \sup_{E[X\xi]=a, X \geq 0} Eu\left(\frac{b}{a}X\right) \\ &\leq \sup_{E[X\xi]=a, X \geq 0} Eu(X) = V(a) < +\infty, \end{aligned}$$

where the first inequality is due to $u(\cdot)$ being increasing. The proof is complete. \square

COROLLARY 3.1 *If $V(a) < +\infty$ for some $a > 0$, then there exists $a_0 > 0$ such that Problem (1.1) admits a unique optimal solution for all $0 < a \leq a_0$.*

Proof: It follows from Theorem 3.1 that there exists λ_0 with $E[(u')^{-1}(\lambda_0\xi)\xi] < +\infty$; consequently the Lagrange multiplier exists for any $0 < a \leq a_0 := E[(u')^{-1}(\lambda_0\xi)\xi]$ by Lemma 2.1. On the other hand, Lemma 3.1 yields that $V(a) < +\infty$ for all a ; hence the desired result follows by virtue of Theorem 1.1. \square

Now let us continue with Example 2.3.

EXAMPLE 3.1 In Example 2.3, take $\lambda = 2$. We have proved that $a_2 := E[(u')^{-1}(2\xi)\xi] < +\infty$. Denote $X^* = (u')^{-1}(2\xi)$. Then

$$\begin{aligned} Eu(X^*) &= Eu(g(2\xi)) \\ &= E[2\xi g(2\xi) + \int_0^{1/(2\xi)} \frac{p(y)}{y^2} dy] \\ &= 2a_2 + \sum_{n=2}^{+\infty} \frac{a_n}{n-1} E[(2\xi)^{-(n-1)}] \\ &= 2a_2 + \sum_{n=2}^{+\infty} \frac{2^{-(n-1)}}{n^2(n-1)} \\ &< +\infty. \end{aligned}$$

Theorem 1.1 suggests that X^* is the unique optimal solution for (1.1) with parameter a_2 and, in particular, $V(a_2) = Eu(X^*) < +\infty$. By Lemma 3.1, we know $V(a) < +\infty$ for any $a > 0$, i.e., (1.1) is well-posed for any $a > 0$.

However, we have proved in Example 2.3 that $E[(u')^{-1}(\lambda\xi)\xi] = a$ admits no solution for any $a > a_1$. Therefore Problem (1.1) with parameter $a > a_1$ is well-posed; yet it admits no optimal solution (i.e., the problem is not attainable).

4 ILL-POSEDNESS WHEN LAGRANGE MULTIPLIER EXISTS

The last section demonstrated that one of the possible consequences of the non-existence of a Lagrange multiplier is the ill-posedness of the underlying optimization problem. This section aims to show via an example that Problem (1.1) may be ill-posed even if the Lagrange multiplier *does* exist for *any* $a > 0$.

EXAMPLE 4.1 Let

$$u(x) = \begin{cases} \sqrt{x}, & 0 \leq x \leq 1, \\ 1 - \ln 2 + \ln(1+x), & x > 1, \end{cases}$$

and ξ be a positive random variable such that $E[\ln \frac{1}{\xi}] = +\infty$. It is easy to check that $u(\cdot)$ has all the required properties, and

$$(u')^{-1}(x) = \begin{cases} \frac{1}{x} - 1, & 0 < x \leq 0.5, \\ \frac{1}{4x^2}, & x > 0.5. \end{cases}$$

Hence

$$f(\lambda) = E[(u')^{-1}(\lambda\xi)\xi] = \frac{1}{\lambda}E[(1 - \lambda\xi)\mathbf{1}_{\lambda\xi \leq 0.5}] + E[\frac{1}{4\lambda^2\xi}\mathbf{1}_{\lambda\xi > 0.5}] \leq \frac{3}{2\lambda} < +\infty \quad \forall \lambda > 0.$$

As a result, the Lagrange multiplier exists for any $a > 0$. However, for any $\lambda > 0$,

$$E[u((u')^{-1}(\lambda\xi))] = E[(1 - \ln 2 - \ln(\lambda\xi))\mathbf{1}_{\lambda\xi \leq 0.5}] + E[\frac{1}{2\lambda\xi}\mathbf{1}_{\lambda\xi > 0.5}] \geq E[\ln \frac{1}{\xi}\mathbf{1}_{\lambda\xi \leq 0.5}] - \ln(\lambda) = +\infty.$$

REMARK 4.1 In existing literature it is usually assumed, either explicitly (see, e.g., Karatzas (1997), p. 37, (2.2.13)) or implicitly, that the problem is well-posed for all a . The preceding example proves that the well-posedness is not guaranteed even when the Lagrange multiplier exists.

5 OPTIMAL SOLUTION

Having discussed on the ill-posedness and non-attainability, we are now in a position to study the optimal solution of (1.1). The problems with Theorem 1.1 are two-fold. On one hand, the required conditions that the Lagrange equation $E[(u')^{-1}(\lambda\xi)\xi] = a$ admits a positive solution and

that $E[u((u')^{-1}(\lambda\xi))] < +\infty$ do not necessarily hold (as already demonstrated), and on the other hand even if the conditions do hold, they are implicit and/or hard to verify. In this section, we will present conditions that are explicit and easy to use.

Recall that $f(\lambda) = E[(u')^{-1}(\lambda\xi)\xi]$, $\lambda > 0$. If $f(\lambda) = +\infty$ for any $\lambda > 0$, then it follows from Theorem 3.1 that $V(a) = +\infty$ for any $a > 0$, which is a pathological case. Hence we assume that there exists a $\lambda > 0$ such that $f(\lambda) < +\infty$. Denote $\lambda_0 = \inf\{\lambda > 0 : f(\lambda) < +\infty\} < +\infty$ and $a_0 = f(\lambda_0+)$ (notice that $a_0 = +\infty$ is possible, and $a_0 = f(\lambda_0)$ when $\lambda_0 > 0$).

PROPOSITION 5.1 *Suppose $\lambda_0 < +\infty$. We have the following conclusions.*

- (i) *If $a_0 < +\infty$, then Problem (1.1) with parameter $a > 0$ admits a unique optimal solution if and only if $E[u((u')^{-1}(\lambda_0\xi))] < +\infty$ and $a \leq a_0$.*
- (ii) *If $a_0 = +\infty$, then Problem (1.1) admits a unique optimal solution for any $a > 0$ if and only if $E[u((u')^{-1}(\xi))] < +\infty$.*

Proof: (i) is clear in view of Theorem 1.1 and Lemma 3.1. To prove (ii), if $a_0 = +\infty$, by Lemma 2.1, $f(\cdot)$ is continuous on $(\lambda_0, +\infty)$ with $f(\lambda_0+) = +\infty$ and $f(+\infty) = 0$; hence the Lagrange multiplier exists for any $a > 0$. Now, if $E[u((u')^{-1}(\xi))] < +\infty$, using $u(x) \geq xu'(x)$ with $x = (u')^{-1}(\xi)$, we have

$$+\infty > E[u((u')^{-1}(\xi))] \geq E[(u')^{-1}(\xi)\xi] =: a_1.$$

It follows from Theorem 1.1 that $V(a_1) = E[u((u')^{-1}(\xi))] < +\infty$. Lemma 3.1 further yields $V(a) < +\infty$, $\forall a > 0$. The desired result is now a consequence of Theorem 1.1. \square

Now we derive some sufficient conditions, explicit in terms of $u(\cdot)$ or ξ , for the existence of a unique optimal solution to (1.1). First we have the following simple case.

THEOREM 5.1 *If $\varepsilon = \text{essinf } \xi > 0$, then Problem (1.1) admits a unique optimal solution for any $a > 0$.*

Proof: Given $a > 0$. For any feasible solution X of Problem (1.1),

$$Eu(X) \leq u(EX) \leq u\left(\frac{E[X\xi]}{\varepsilon}\right) = u\left(\frac{a}{\varepsilon}\right).$$

Therefore $V(a) < +\infty$.

Meanwhile, for any $\lambda > 0$,

$$f(\lambda) = E[(u')^{-1}(\lambda\xi)\xi] \leq \frac{1}{\lambda} E[u((u')^{-1}(\lambda\xi))] \leq \frac{1}{\lambda} u((u')^{-1}(\lambda\varepsilon)) < +\infty.$$

This proves the existence of the Lagrange multiplier $\lambda > 0$ for any $a > 0$. By Theorem 1.1, $X_\lambda = (u')^{-1}(\lambda\xi)$ is the unique optimal solution for (1.1). \square

Let us make some preparations for our main result.

Define $R(x) = -\frac{xu''(x)}{u'(x)} \geq 0$ as the *Arrow-Pratt index of risk aversion* of the utility function $u(\cdot)$.

LEMMA 5.1 *If $\liminf_{x \rightarrow +\infty} R(x) > 0$, then $\limsup_{x \rightarrow +\infty} \frac{u'(kx)}{u'(x)} < 1$ for any $k > 1$.*

Proof: Because $\liminf_{x \rightarrow +\infty} R(x) > 0$, there exist $M > 0$, $K > 0$, such that $R(x) \geq K$ for any $x \geq M$. For any $x \geq M$, $k > 1$,

$$\begin{aligned}
\frac{u'(kx)}{u'(x)} - 1 &= \frac{u'(kx) - u'(x)}{u'(x)} \\
&= \frac{\int_x^{kx} u''(y) dy}{u'(x)} \\
&= -\frac{\int_x^{kx} R(y) u'(y)/y dy}{u'(x)} \\
&\leq -\frac{\int_x^{kx} R(y) u'(kx)/y dy}{u'(x)} \\
&= -\frac{u'(kx)}{u'(x)} \int_x^{kx} R(y)/y dy \\
&\leq -\frac{u'(kx)}{u'(x)} K \int_x^{kx} 1/y dy \\
&= -\frac{u'(kx)}{u'(x)} K \ln k.
\end{aligned}$$

Therefore $\frac{u'(kx)}{u'(x)} \leq \frac{1}{1+K \ln k}$ which implies $\limsup_{x \rightarrow +\infty} \frac{u'(kx)}{u'(x)} \leq \frac{1}{1+K \ln k} < 1$. \square

LEMMA 5.2 $\limsup_{x \rightarrow 0+} \frac{(u')^{-1}(\lambda x)}{(u')^{-1}(x)} < +\infty$ for any $0 < \lambda < 1$ if and only if $\limsup_{x \rightarrow +\infty} \frac{u'(kx)}{u'(x)} < 1$ for any $k > 1$.

Proof: We first claim that $\limsup_{x \rightarrow 0+} \frac{(u')^{-1}(\lambda x)}{(u')^{-1}(x)} < +\infty$ for any $0 < \lambda < 1$ if and only if $\exists 0 < \bar{\lambda} < 1$ such that $\limsup_{x \rightarrow 0+} \frac{(u')^{-1}(\bar{\lambda} x)}{(u')^{-1}(x)} < +\infty$.

To prove this claim, suppose $\limsup_{x \rightarrow 0+} \frac{(u')^{-1}(\bar{\lambda} x)}{(u')^{-1}(x)} < +\infty$ for some $0 < \bar{\lambda} < 1$. Then

$$\begin{aligned}
&\limsup_{x \rightarrow 0+} \frac{(u')^{-1}(\bar{\lambda}^2 x)}{(u')^{-1}(x)} \\
&= \limsup_{x \rightarrow 0+} \frac{(u')^{-1}(\bar{\lambda}^2 x)}{(u')^{-1}(\bar{\lambda} x)} \frac{(u')^{-1}(\bar{\lambda} x)}{(u')^{-1}(x)} \\
&\leq \limsup_{x \rightarrow 0+} \frac{(u')^{-1}(\bar{\lambda}^2 x)}{(u')^{-1}(\bar{\lambda} x)} \limsup_{x \rightarrow 0+} \frac{(u')^{-1}(\bar{\lambda} x)}{(u')^{-1}(x)} \\
&< +\infty.
\end{aligned}$$

From induction it follows $\limsup_{x \rightarrow 0+} \frac{(u')^{-1}(\bar{\lambda}^n x)}{(u')^{-1}(x)} < +\infty$ for any $n \in \mathbb{N}$. Since $\limsup_{x \rightarrow 0+} \frac{(u')^{-1}(\lambda x)}{(u')^{-1}(x)}$ is non-increasing in λ , $\limsup_{x \rightarrow 0+} \frac{(u')^{-1}(\lambda x)}{(u')^{-1}(x)} < +\infty$ for any $0 < \lambda < 1$.

Similarly, one can prove that $\limsup_{x \rightarrow +\infty} \frac{u'(kx)}{u'(x)} < 1$ for any $k > 1$ if and only if $\exists \bar{k} > 1$ such that $\limsup_{x \rightarrow +\infty} \frac{u'(\bar{k}x)}{u'(x)} < 1$.

Now, suppose $L = \limsup_{x \rightarrow 0+} \frac{(u')^{-1}(\frac{1}{2}x)}{(u')^{-1}(x)} < +\infty$ (notice that $L \geq 1$). Then there exists $\delta > 0$ such that for any $x \in (0, \delta]$,

$$\begin{aligned} & \frac{(u')^{-1}(\frac{1}{2}x)}{(u')^{-1}(x)} \leq 2L \\ \Rightarrow & \frac{1}{2}x \geq u'(2L(u')^{-1}(x)) \\ \Rightarrow & \frac{1}{2} \geq \frac{u'(2L(u')^{-1}(x))}{u'((u')^{-1}(x))} \\ \Rightarrow & \frac{u'(2Ly)}{u'(y)} \leq \frac{1}{2}, \quad \forall y \geq (u')^{-1}(\delta) \\ \Rightarrow & \limsup_{x \rightarrow +\infty} \frac{u'(2Lx)}{u'(x)} \leq \frac{1}{2}. \end{aligned}$$

Therefore $\limsup_{x \rightarrow +\infty} \frac{u'(kx)}{u'(x)} < 1$ for any $k > 1$.

The proof for the other direction is similar. □

Recall that we have defined $f(\lambda) = E[(u')^{-1}(\lambda\xi)\xi]$ and $\lambda_0 = \inf\{\lambda > 0 : f(\lambda) < +\infty\}$.

PROPOSITION 5.2 *Suppose one of the following conditions is satisfied:*

- (i) $\liminf_{x \rightarrow +\infty} R(x) > 0$.
- (ii) $\limsup_{x \rightarrow +\infty} \frac{u'(kx)}{u'(x)} < 1$ for some $k > 1$.
- (iii) $\limsup_{x \rightarrow 0} \frac{(u')^{-1}(\lambda x)}{(u')^{-1}(x)} < +\infty$ for some $\lambda \in (0, 1)$.

Then the Lagrange multiplier exists for any $a > 0$ if and only if $\lambda_0 < +\infty$.

Proof: The necessity is obvious. To prove the sufficiency, note that if $\lambda_0 < +\infty$, then there exists $\lambda_1 > 0$ such that $f(\lambda_1) < +\infty$, which by the monotonicity of $f(\cdot)$ further implies that $f(\lambda) < +\infty$ $\forall \lambda > \lambda_1$. For any $\lambda \in (0, \lambda_1]$, denote $k = \lambda/\lambda_1 \in (0, 1]$.

Since one of the three given conditions is satisfied, by Lemmas 5.1 and 5.2 it must have $1 \leq L = \limsup_{x \rightarrow 0} \frac{(u')^{-1}(kx)}{(u')^{-1}(x)} < +\infty$. Hence there exists $\delta > 0$ such that $\frac{(u')^{-1}(kx)}{(u')^{-1}(x)} < 2L$ for any $x \in (0, \lambda_1\delta]$. Now, for any $\lambda > 0$,

$$\begin{aligned} E[(u')^{-1}(\lambda\xi)\xi \mathbf{1}_{\xi \leq \delta}] &= E \left[\frac{(u')^{-1}(\lambda\xi)}{(u')^{-1}(\lambda_1\xi)} (u')^{-1}(\lambda_1\xi)\xi \mathbf{1}_{\xi \leq \delta} \right] \\ &\leq 2LE[(u')^{-1}(\lambda_1\xi)\xi \mathbf{1}_{\xi \leq \delta}] \\ &\leq 2Lf(\lambda_1), \\ E[(u')^{-1}(\lambda\xi)\xi \mathbf{1}_{\xi > \delta}] &= \frac{1}{\lambda} E[(u')^{-1}(\lambda\xi)(\lambda\xi) \mathbf{1}_{\xi > \delta}] \\ &\leq \frac{1}{\lambda} E[u((u')^{-1}(\lambda\xi)) \mathbf{1}_{\xi > \delta}] \\ &\leq \frac{1}{\lambda} u((u')^{-1}(\lambda\delta)). \end{aligned}$$

Hence,

$$\begin{aligned}
f(\lambda) &= E[(u')^{-1}(\lambda\xi)\xi] \\
&= E[(u')^{-1}(\lambda\xi)\xi\mathbf{1}_{\xi\leq\delta}] + E[(u')^{-1}(\lambda\xi)\xi\mathbf{1}_{\xi>\delta}] \\
&\leq 2Lf(\lambda_1) + \frac{1}{\lambda}u((u')^{-1}(\lambda\delta)) \\
&< +\infty.
\end{aligned}$$

This shows that in fact $\lambda_0 = 0$, and hence the equation $f(\lambda) = a$ admits a positive solution $\lambda(a)$ for any $a > 0$. \square

REMARK 5.1 The preceding proof also shows that under the condition of Proposition 5.2, the following claims are equivalent:

- (i) The Lagrange multiplier exists for any $a > 0$.
- (ii) $\lambda_0 < +\infty$.
- (iii) $\lambda_0 = 0$.
- (iv) $f(1) < +\infty$.
- (v) $f(\lambda) < +\infty \forall \lambda > 0$.

THEOREM 5.2 *Under the condition of Proposition 5.2, Problem (1.1) admits a unique optimal solution for any $a > 0$ if and only if $E[u((u')^{-1}(\xi))] < +\infty$.*

Proof: It suffices to prove the sufficiency. If $E[u((u')^{-1}(\xi))] < +\infty$, then $f(1) = E[(u')^{-1}(\xi)\xi] \leq E[u((u')^{-1}(\xi))] < +\infty$. Thus $\lambda_0 = 0$ and $a_0 = f(\lambda_0+) = +\infty$. It follows from Proposition 5.1 then that Problem (1.1) admits a unique optimal solution. \square

The conditions in the preceding theorem, $\liminf_{x \rightarrow +\infty} -\frac{xu''(x)}{u'(x)} \geq 0$ and $E[u((u')^{-1}(\xi))] < +\infty$, are very easy to verify. For example, a commonly used utility function is $u(x) = x^\alpha$, $0 < \alpha < 1$. The two conditions are satisfied when ξ is lognormal.

REMARK 5.2 Example 3.1 shows that the conclusion of Theorem 5.2 can be false in the absence of its condition.

COROLLARY 5.1 *If $E[\xi^{-\alpha}] < +\infty \forall \alpha \geq 1$, then, under the condition of Proposition 5.2, Problem (1.1) admits a unique optimal solution for any $a > 0$.*

Proof: It suffices to prove that $E[u((u')^{-1}(\xi))] < +\infty$ holds automatically. Under the condition of Proposition 5.2, there is $L \geq 2$ such that $(u')^{-1}(x) < L(u')^{-1}(2x) \forall x \in (0, 1)$. Denote $L_0 = \sup_{x \in [\frac{1}{2}, 1]} (u')^{-1}(x) < +\infty$. For any $x \in (0, 1)$, find $n \in \mathbb{N}$ so that $\frac{1}{2} \leq 2^n x < 1$. Then $(u')^{-1}(x) < L(u')^{-1}(2x) < L^2(u')^{-1}(2^2x) < \dots < L^n(u')^{-1}(2^n x) \leq L^n L_0 \leq L^{-\log_2 x} L_0 = x^{-\log_2 L} L_0$. By virtue of the fact that $u'(+\infty) = 0$, we may assume that $u(x) \leq L_1 x \forall x \geq (u')^{-1}(1)$. Therefor for any $x \in (0, 1)$, we have $u((u')^{-1}(x)) \leq L_1(u')^{-1}(x) < L_0 L_1 x^{-\log_2 L}$. Finally, $E[u((u')^{-1}(\xi))] \leq E[u((u')^{-1}(\xi))\mathbf{1}_{\xi < 1}] + u((u')^{-1}(1)) \leq L_0 L_1 E[\xi^{-\log_2 L}] + u((u')^{-1}(1)) < +\infty$. \square

REMARK 5.3 If ξ is lognormal, then the assumption that $E[\xi^{-\alpha}] < +\infty \forall \alpha \geq 1$ holds automatically. (In the context of portfolio selection with the prices of the underlying stocks following geometric Brownian motion, ξ is typically a lognormal random variable – under certain conditions of course; for details see Remark 3.1 in Beliecki et. al (2005).) On the other hand, this assumption could be weakened to that $E[\xi^{-\alpha_0}] < +\infty$ for *certain* α_0 (the value of which could be precisely given). We leave the details to the interested readers.

Recall that in Section 4 we presented an example where Problem (1.1) is ill-posed even though the Lagrange multiplier exists for any $a > 0$. The following result shows that this will not occur for certain ξ .

Let $F(\cdot)$ be the probability distribution function of ξ . In view of Theorem 5.1, we assume $\text{essinf } \xi = 0$, which in turn ensures $F(x) > 0 \forall x > 0$.

THEOREM 5.3 *If $\liminf_{x \rightarrow 0} \frac{x F'(x)}{F(x)} > 0$, and $E[(u')^{-1}(\lambda \xi) \xi] = a > 0$ for some $\lambda > 0$, then Problem (1.1) with parameter a is well-posed and admits a unique optimal solution.*

Proof: Since $\liminf_{x \rightarrow 0} \frac{x F'(x)}{F(x)} > 0$, there exist $M > 0$ and $K > 0$ such that $\frac{x F'(x)}{F(x)} \geq \frac{1}{K}$ for any $0 < x \leq M$. Then

$$\begin{aligned}
& E[u((u')^{-1}(\lambda \xi)) \mathbf{1}_{\xi < M}] \\
&= \int_0^M u((u')^{-1}(\lambda x)) dF(x) \\
&= \int_0^M \int_M^x du((u')^{-1}(\lambda y)) dF(x) + \int_0^M u((u')^{-1}(\lambda M)) dF(x) \\
&= \lambda \int_0^M \int_M^x y d[(u')^{-1}(\lambda y)] dF(x) + u((u')^{-1}(\lambda M)) F(M) \\
&= \lambda \int_0^M \left(x(u')^{-1}(\lambda x) - M(u')^{-1}(\lambda M) + \int_x^M (u')^{-1}(\lambda y) dy \right) dF(x) \\
&\quad + u((u')^{-1}(\lambda M)) F(M) \\
&= \lambda \int_0^M x(u')^{-1}(\lambda x) dF(x) + \lambda \int_0^M \int_x^M (u')^{-1}(\lambda y) dy dF(x) \\
&\quad + [u((u')^{-1}(\lambda M)) - \lambda M(u')^{-1}(\lambda M)] F(M) \\
&= \lambda \int_0^M x(u')^{-1}(\lambda x) dF(x) + \lambda \int_0^M \int_0^y dF(x) (u')^{-1}(\lambda y) dy \\
&\quad + [u((u')^{-1}(\lambda M)) - \lambda M(u')^{-1}(\lambda M)] F(M) \\
&= \lambda \int_0^M x(u')^{-1}(\lambda x) dF(x) + \lambda \int_0^M F(y) (u')^{-1}(\lambda y) dy \\
&\quad + [u((u')^{-1}(\lambda M)) - \lambda M(u')^{-1}(\lambda M)] F(M) \\
&\leq \lambda \int_0^M x(u')^{-1}(\lambda x) dF(x) + K \lambda \int_0^M y F'(y) (u')^{-1}(\lambda y) dy \\
&\quad + [u((u')^{-1}(\lambda M)) - \lambda M(u')^{-1}(\lambda M)] F(M) \\
&\leq \lambda(1 + K)a + [u((u')^{-1}(\lambda M)) - \lambda M(u')^{-1}(\lambda M)] F(M) \\
&< +\infty.
\end{aligned}$$

Consequently,

$$\begin{aligned}
& E[u((u')^{-1}(\lambda\xi))] \\
&= E[u((u')^{-1}(\lambda\xi))\mathbf{1}_{\xi < M}] + E[u((u')^{-1}(\lambda\xi))\mathbf{1}_{\xi \geq M}] \\
&\leq E[u((u')^{-1}(\lambda\xi))\mathbf{1}_{\xi < M}] + u((u')^{-1}(\lambda M)) \\
&< +\infty.
\end{aligned}$$

The desired result follows then from Theorem 1.1. \square

REMARK 5.4 The condition $\liminf_{x \rightarrow 0} \frac{x F'(x)}{F(x)} > 0$ implicitly requires that $F(\cdot)$ be differentiable in the neighborhood of 0. Notice that this requirement is purely technical so as to make the result neater. One could replace the condition $\liminf_{x \rightarrow 0} \frac{x F'(x)}{F(x)} > 0$ by a weaker one without having to assume the differentiability of $F(\cdot)$ (as hinted by the preceding proof – the details are left to the interested reader). On the other hand, the condition is satisfied if ξ is lognormal.

Combining Theorem 1.1 and Theorem 5.3, we have immediately

COROLLARY 5.2 *Suppose $\liminf_{x \rightarrow 0} \frac{x F'(x)}{F(x)} > 0$. Then Problem (1.1) with parameter $a > 0$ admits an optimal solution if and only if the Lagrange multiplier λ exists corresponding to a , in which case the unique optimal solution is $X^* = (u')^{-1}(\lambda\xi)$.*

The following synthesized result gives easily verifiable conditions under which Problem (1.1) is completely solved.

THEOREM 5.4 *We have the following conclusions.*

(i) *If $\liminf_{x \rightarrow +\infty} \left(-\frac{x u''(x)}{u'(x)}\right) > 0$, then the following statements are equivalent:*

- (ia) *Problem (1.1) is well-posed for any $a > 0$.*
- (ib) *Problem (1.1) admits a unique optimal solution.*
- (ic) *$E[u((u')^{-1}(\xi))] < +\infty$.*
- (id) *$\exists \lambda > 0$ such that $E[u((u')^{-1}(\lambda\xi))] < +\infty$.*

Moreover, when one of (ia)–(id) holds the optimal solution to (1.1) with parameter $a > 0$ is $X^ = (u')^{-1}(\lambda(a)\xi)$, where $\lambda(a)$ is the Lagrange multiplier corresponding to a .*

(ii) *If $\limsup_{x \rightarrow 0} \left(-\frac{x F'(x)}{F(x)}\right) < 0$, then Problem (1.1) is well-posed for any $a > 0$ if and only if $E[(u')^{-1}(\lambda\xi)\xi] < +\infty$ for some $\lambda > 0$, in which case there exists $0 < a_0 \leq +\infty$ so that (1.1) admits a unique optimal solution $X^* = (u')^{-1}(\lambda(a)\xi)$ for any $a > 0$ (if $a_0 = +\infty$) or for any $0 < a \leq a_0$ (if $a_0 < +\infty$).*

Proof: (i) If (1.1) is well-posed for any $a > 0$, then Theorem 3.1 yields that $f(\lambda_0) < +\infty$ for some $\lambda_0 > 0$. It follows from Proposition 5.2 and Theorem 1.1 that (1.1) admits a unique optimal solution for any $a > 0$. The desired equivalence is then a consequence of Theorem 5.2 and Theorem 1.1.

(ii) The first conclusion (“if and only if”) follows from Theorems 3.1 and 5.3. For the second conclusion, let $\lambda_0 = \inf\{\lambda > 0 : f(\lambda) < +\infty\} < +\infty$ and $a_0 = f(\lambda_0+)$. Then the Lagrange multiplier exists for any $a > 0$ (if $a_0 = +\infty$) or for any $0 < a \leq a_0$ (if $a_0 < +\infty$), and Corollary 5.2 completes the proof. \square

REMARK 5.5 Portfolio selection is essentially an endeavor that an investor, given a market (represented by ξ or its distribution function $F(\cdot)$), tries to make the best out of his initial wealth (namely a) taking advantage of the availability of the market, where the “best” is measured by her preference (i.e. the utility function $u(\cdot)$). We have shown that these entities, namely $F(\cdot)$, a , and $u(\cdot)$, must coordinate well, otherwise one may end up with a wrong model. The assumptions stipulated in Theorem 5.4 tell precisely how this well-coordination can be translated into mathematical conditions.

6 CONCLUDING REMARKS

The stochastic optimization problem studied in this paper, though interesting in its own right, has profound applications in financial asset allocation among others. It is demonstrated that many assumptions that have been taken for granted, such as the well-posedness of the problem, existence of the Lagrange multiplier, and existence of an optimal solution, may be invalid in the first place. In particular, the issue of well-posedness is equally important, if not more important, than that of finding an optimal solution from a modeling point of view. Attainability of optimal solutions is another important matter: if an optimal solution is not attainable, as is the case with Example 3.1, then one has to resort to finding an asymptotically optimal solution. Mathematically, both the ill-posedness and the non-attainability are symptomized by the non-existence of the Lagrange multiplier, as analyzed in details in this paper.

It is worth noting that the results of this paper have been utilized in solving a sub-problem of the continuous-time behavioral portfolio selection model Jin and Zhou (2006), where the ill-posedness is more a rule than an exception.

REFERENCES

- T.R. BIELECKI, H. JIN, S.R. PLISKA AND X.Y. ZHOU (2005): Continuous-time mean–variance portfolio selection with bankruptcy prohibition, *Math. Finance* 15, pp. 213-244.
- J. CVITANIC AND I. KARATZAS(1992), Convex duality in constrained portfolio optimization, *Ann. Appl. Probab.* 2, pp. 767-818.
- I. KARATZAS (1997), *Lectures on the Mathematics of Finance*, American Mathematical Society.
- I. KARATZAS AND S.E. SHREVE (1998), *Methods of Mathematical Finance*, Springer-Verlag, New York.
- R. KORN (1997), *Optimal Portfolios*, World Scientific, Singapore.
- R. KORN AND H. KRAFT (2004), On the stability of continuous-time portfolio problems with stochastic opportunity set, *Math. Finance* 14, pp. 403-414.
- H. JIN AND X.Y. ZHOU (2006), Behavioral portfolio selection in continuous time, Working paper, Department of Systems Engineering and Engineering Management, The Chinese University of Hong Kong.